Axisymmetric Ideal MHD Tokamak Equilibria

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Abstract

These notes are about the Grad-Shafranov equation, which is derived from the static ideal MHD equations assuming axisymmetry. Selected aspects of the equation and its analytical solutions are discussed.

Contents

1 Grad-Shafranov equation

1.1 Setting

The considered geometry is that of Fig. 1 in which a cross-section of an axisymmetric toroidal tokamak is shown. The axis of symmetry is the Z-axis and the ignorable angle is ϕ. The R coordinate is a radial coordinate and R_0 and a are called the major axis and minor axis of the tokamak. The ratio $\epsilon \equiv a/R_0$ is the inverse aspect ratio, κ the elongation and δ the triangularity.

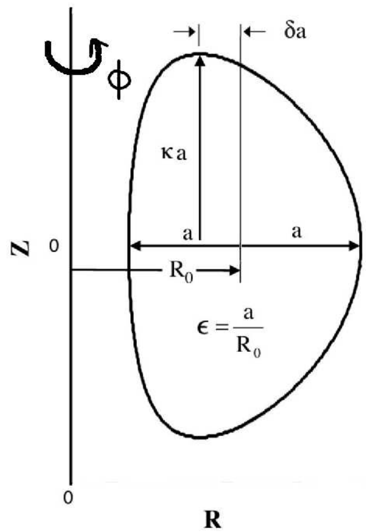

Figure 1: The considered geometry.

Plasma floats in a somewhat D-shaped configuration surrounded by a conducting wall. Between the hot plasma and the wall is a vacuum such that the plasma has to be kept in place with external magnetic fields. Superconducting magnets produce a large toroidal (in the ϕ-direction) magnetic field $B_\phi \hat{\boldsymbol{\phi}}$ (a hat is used to denote a unit vector) that, combined with poloidal (in the cross-sectional plane) electric currents $\mathbf{J}_p$ inside the plasma, produce an inward Lorentz force $\mathbf{J} \times \mathbf{B}$ counteracting the outward pressure force. Moreover a toroidal electric current $J_\phi \hat{\boldsymbol{\phi}}$ is generated in the plasma which, via Ampere's law $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$, produces a poloidal magnetic field also constituting an inward Lorentz force.

The relevant static magnetohydrodynamic equations are the balance between the pressure force and the Lorentz force, Ampere's law and the fact that the magnetic field $\mathbf{B}$ is divergence-free (by a proper rescaling, the constant μ_0 can be left out everywhere in the following):

$$\nabla p = \mathbf{J} \times \mathbf{B} \ , \ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \ , \ \nabla \cdot \mathbf{B} = 0 \tag{1}$$

1.2 Derivation

We introduce the cylindrical coordinate system (R, Z, ϕ) shown in Fig. 1 with the origin at the centre of the tokamak. The magnetic field can then be written in terms of a magnetic vector potential $\mathbf{A}$ as

$$\mathbf{B} \equiv \nabla \times \mathbf{A} = -\frac{\partial A_\phi}{\partial z}\hat{\mathbf{R}} + \left(\frac{\partial A_R}{\partial z} - \frac{\partial A_z}{\partial R}\right)\hat{\boldsymbol{\phi}} + \frac{1}{R}\frac{\partial R A_\phi}{\partial \phi}\hat{\boldsymbol{Z}} \tag{2}$$

We split $\mathbf{J}$ and $\mathbf{B}$ in a poloidal and toroidal component using $\nabla\phi = \hat{\boldsymbol{\phi}}/R$

$$\mathbf{B} = \nabla\psi \times \nabla\phi + B_\phi \hat{\boldsymbol{\phi}} \tag{3}$$

$$\mathbf{J} = \nabla F \times \nabla\phi + J_\phi \hat{\boldsymbol{\phi}} \tag{4}$$

With $\nabla\psi \times \nabla\phi = \nabla \times (\psi\nabla\phi)$ and $\partial_\phi = 0$, the above formulation manifestly satisfies $\nabla \cdot \mathbf{B} = 0$. Furthermore, with a subscript p denoting the poloidal components

$$\int \mathbf{B}_p \cdot d\mathbf{S} = \oint_{\partial S} \psi \nabla\phi \cdot d\mathbf{l} = \oint_{\partial S} \psi \cdot d\phi = 2\pi\Delta\psi \tag{5}$$

showing that $2\pi\psi$ represents the poloidal magnetic flux. The magnetic field lines therefore lie on surfaces of constant ψ, i.e. $\mathbf{B} \cdot \nabla\psi = 0$. Furthermore from $\mathbf{J} \times \mathbf{B} = \nabla p$ we have $\mathbf{B} \cdot \nabla p = 0$ such that $p = p(\psi)$ and $\mathbf{J} \cdot \nabla p = \nabla F \times \nabla p = 0$ such that $F = F(\psi)$. Note that $\mathbf{B}_p = \nabla \times ((\psi/R)\hat{\boldsymbol{\phi}})$ such that ψ is related to the magnetic vector potential $\mathbf{A}$. Analogously $\mathbf{J}_p = \nabla \times ((F/R)\hat{\boldsymbol{\phi}})$ such that F is related to the 'current vector potential' $\mathbf{B}/\mu_0$

$$\psi = R A_\phi \tag{6}$$

$$F = R B_\phi/\mu_0 \tag{7}$$

The static force balance yields with the proposed split in poloidal and toroidal components

$$\mathbf{J}_p \times B_\phi \hat{\boldsymbol{\phi}} + J_\phi \hat{\boldsymbol{\phi}} \times \mathbf{B}_p = \nabla p \tag{8}$$

After substitution and noticing $\hat{\boldsymbol{\phi}} \cdot \nabla\psi = \hat{\boldsymbol{\phi}} \cdot \nabla F = 0$

$$-\frac{B_\phi}{R}\nabla F + \frac{J_\phi}{R}\nabla\psi = \nabla p \tag{9}$$

With $\nabla F(\psi) = \frac{dF}{d\psi}\nabla\psi$ and $\nabla p(\psi) = \frac{dp}{d\psi}\nabla\psi$ this yields after multiplying with R

$$J_\phi = R\frac{dp}{d\psi} + B_\phi \frac{dF}{d\psi} \tag{10}$$

With $B_\phi = \mu_0 F/R$ the first term becomes $\mu_0 F dF/d\psi$ leaving only J_ϕ unspecified. From Ampere's law $\mu_0 \mathbf{J} = \nabla \times \nabla \times \mathbf{A} = -\nabla^2 \mathbf{A}$ such that

$$-\nabla^2 \mathbf{A} = -\nabla^2 A_\phi - A_\phi/R^2 = -\nabla^2(\psi/R) + \psi/R^3 \tag{11}$$

where use has been made of the non-zero derivatives of cylindrical unit vectors $\partial\hat{\mathbf{R}}/\partial\phi = \hat{\boldsymbol{\phi}}$ and $\partial\hat{\boldsymbol{\phi}}/\partial\phi = -\hat{\mathbf{R}}$ which combine to give $\nabla^2_\phi \hat{\phi} = R^{-2}\frac{\partial^2 \hat{\phi}}{\partial \phi^2} = -\hat{\boldsymbol{\phi}}/R^2$. Therefore

$$-\mu_0 J_\phi = \frac{1}{R}\left(\frac{\partial^2\psi}{\partial R^2} + \frac{\partial^2\psi}{\partial Z^2}\right) + \frac{1}{R^2}\frac{\partial\psi}{\partial R} = \frac{\partial}{\partial R}\frac{1}{R}\frac{\partial\psi}{\partial R} + \frac{\partial^2\psi}{\partial z^2} = R\nabla\cdot\left(\frac{\nabla\psi}{R^2}\right) \tag{12}$$

and inserting in Eq. 10 we obtain the Grad-Shafranov equation

$$R^2\nabla\cdot\left(\frac{\nabla\psi}{R^2}\right) = -\mu_0 R^2\frac{\partial p}{\partial\psi} - F\frac{dF}{d\psi} \tag{13}$$

1.3 Some properties of the GS-equation

With the definition $\Delta^+ \equiv \frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial z^2}$, the elliptic Grad-Shafranov operator $\Delta^*\psi \equiv R^2\nabla\cdot\left(\nabla\psi/R^2\right)$ satisfies $\Delta^* = \Delta^+ - \frac{1}{R}\frac{\partial}{\partial R}$ and with the Laplacian $\Delta = \Delta^+ + \frac{1}{R}\frac{\partial}{\partial R}$ we have $\Delta^* = \Delta - \frac{2}{R}\frac{\partial}{\partial R}$. The Grad-Shafranov equation $\Delta^*\psi = -\mu_0 R J_\phi(\psi, R)$, can be a linear or non-linear partial differential equation depending on the 'source' RJ_ϕ.

Nondimensionalizing the coordinates $r = R/a$ and $y = Z/a$, introducing a unit flux variable $\bar{\psi} = \psi/\psi_1$ and scaling the source terms with the vacuum magnetic field B_0 at $R = R_0$, i.e. $p' = p/B_0^2$ and $\frac{1}{2}f'^2 = \frac{1}{2}(F^2 - F_\infty^2)/B_0^2 a^2$ with $F_\infty \equiv R_0 B_0$ the Grad-Shafranov equation can be written in dimensionless form

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\bar{\psi}}{\partial r}\right) + \frac{\partial^2\bar{\psi}}{\partial z^2} = -\alpha^2\frac{d\frac{1}{2}f'^2}{d\bar{\psi}} - r^2\alpha^2\frac{dp'}{d\bar{\psi}} \tag{14}$$

The only dimensionless number $\alpha^2 = (B_0 a^2/\psi_1)^2$ can be incorporated into p' and f' and as such can be used to rescale the solution.

For tokamak applications one typically is interested in equilibria with one or more elliptic fixed- or stationary-points (where $\nabla\psi = 0$). In modern tokamaks, field coils are used to produce two hyperbolic fixed-points at the plasma-vacuum interface which are connected with a separatrix separating the plasma and the vacuum regions. These elliptic and hyperbolic points can be seen from Fig. 2 as the O- and X-shaped stationary points respectively.

We finally note an analogy between the Grad-Shafranov equation and classical fluid mechanics, where the Stokes stream function ψ can be used to write the fluid vorticity as $-\Delta^*\psi/R$.

2 Vacuum

2.1 General vacuum solution

Outside the plasma there is zero pressure and current such that $p = F = 0$ and ψ satisfies the vacuum equation

$$\Delta^* \psi = 0 \tag{15}$$

The vacuum field has to 'match' with the plasma solution in a continuous way. Via these matching conditions, the value of ψ outside the plasma is determined on the one hand by the current J_ϕ in the plasma, and on the other hand by the external field coils that might be present.

A general solution is obtained by separation of variables $\psi = f(R)g(Z)$ yielding (complex) exponentials for $g(Z)$. Usually one only retains the top-down symmetric cosine terms for the z-dependence, i.e. $g(Z) = \cos kZ$. The radial equation yields Bessel's equation for $A_\phi = \psi/R$ so that $f(R) = R\,(c_1 J_1(kR) + c_2 Y_1(kR))$. Solutions with different values of k can be superimposed to yield the general vacuum solution

$$\psi_{vac} = \sum_n c_n R \cos(k_n Z)\,(J_1(k_n R) + d_n Y_1(k_n R)) \tag{16}$$

2.2 Vacuum solution of a circular current loop

Noting that the Green's function of the Grad-Shafranov operator Δ^* is given by

$$G(R,Z;R_0,Z_0) = -\frac{1}{2\pi}\sqrt{RR_0}\frac{1}{k}\left[(2-k^2)K(k) - 2E(k)\right] \tag{17}$$

with $k = 4RR_0/((R+R_0)^2 + (Z-Z_0)^2)$ one can easily solve the GS-equation for a circular current loop $\mathbf{J}(R,Z) = I\delta^2(R-a,0)\hat{\boldsymbol{\phi}}$ in the horizontal plane to obtain with $S \equiv \mu_0 I/4\pi$ for the magnetic vector potential $\mathbf{A} = A_\phi \hat{\boldsymbol{\phi}} = \psi/R$ where

$$\frac{\psi(R,\theta)}{R} = SR\frac{4(2-k\,(R,\theta)^2)K(k\,(R,\theta)^2) - 2E(k\,(R,\theta)^2)}{\sqrt{a^2 + R^2 + 2aR\sin(\theta)}} \tag{18}$$

and

$$k\,(R,\theta) = \frac{4aR\sin(\theta)}{a^2 + R^2 + 2aR\sin(\theta)} \tag{19}$$

With $\hat{\mathbf{m}} \equiv \pi a^2 I\hat{\mathbf{Z}}$ the vector potential for $R \gg a$ becomes approximately that of a dipole, i.e. $\mathbf{A} = \frac{\mu_0}{4\pi}\frac{\hat{\mathbf{m}}\times\hat{\mathbf{r}}}{r^2}$ with r a radial coordinate from the centre of the current loop. With $(\hat{\mathbf{Z}} \times \hat{\mathbf{r}}) \cdot \hat{\boldsymbol{\phi}} = r/R$ one thus finds that for $R \gg a$ the poloidal magnetic flux goes to zero as $\psi = A_\phi R = S\pi a^2/r$, consistent with the notion that the magnetic field of a dipole decays with $1/r^2$ and the flux is proportional to the magnetic field times the intersected area. This far-field condition might be suitable for describing astrophysical plasmas with the GS-equation.

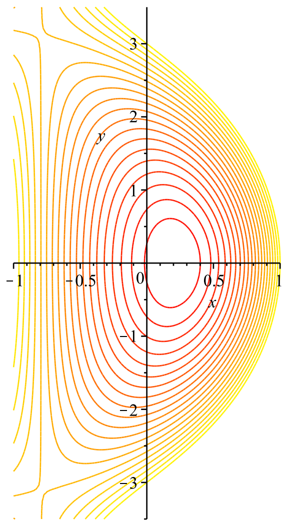

Figure 2: An example of a solution to the Grad-Shafranov equation, given by isocontours of ψ. Two vertically separated X-shaped hyperbolic stationary points are connected by a separatrix, phsyically representing the 'last closed flux surface' separating the plasma region from the vacuum region. An O-shaped elliptic stationary point inside the plasma is located at what is called the magnetic axis.

3 Boundary Conditions

From a practical perspective several types of boundary conditions can be used [4]

- Fixed boundary: the plasma-vacuum boundary is replaced by the surface of a perfect conductor and only the plasma region is calculated, using as a plasma boundary condition $J_\phi = 0$ such that at this boundary $\psi - \psi_{vac} = 0$
- Free boundary: the value of ψ is specified on a closed curve, usually in the vacuum region. The boundary then follows from a certain iso-contour of ψ after having obtained a the solution
- Semi-fixed boundary: Several points on the plasma boundary are prescribed
- Constrained boundary: the equilibrium is solved for a given external magnetic field and some constraint, e.g. a contact point with the domain boundary.

One often uses the boundary value problem, formulated in terms of a radial coordinate r emanating from $R = R_0$ and an angle θ with the R or Z-axis. Anticipating a single elliptic fixed point, located at what is called the magnetic axis, the O-shaped stationary point in Fig. 2, it is convenient to make use of the arbitrariness of a constant shift in ψ to set it to zero there. One often also sets ψ to 1 at the boundary:

$$\psi = 1 \text{ at } r = f(\theta), \text{ the plasma boundary} \tag{20}$$

$$\psi = \psi_R = \psi_Z = 0 \text{ at } R = R_m, Z = 0, \text{ the `magnetic axis'} \tag{21}$$

This redundancy in the number of boundary conditions fixes the magnitude (not the shape) of RJ_ϕ. The relative magnitude of $\partial\frac{1}{2}F^2/\partial\psi$ compared to $R^2\partial p/\partial\psi$ is in turn fixed as well, when one fixes the position $R = R_m$ of the magnetic axis.

4 Analytical solutions

The several analytical solutions that exist depend on specific choices for p and F which occur as 'source terms' on the right hand side of the GS-equation

$$\Delta^* \psi = -\mu_0 R J_\phi = -R^2 \frac{\partial p}{\partial \psi} - F \frac{\partial F}{\partial \psi} \tag{22}$$

Given a general solution the plasma boundary can be obtained afterwards by taking a specific iso-contour of ψ. The vacuum solution then has to be 'matched' to this solution. Alternatively a fixed boundary can be used on which the current density J_ϕ vanishes, thereby physically constituting the effect of a perfectly conducting wall. For mathematical convenience this boundary is usually taken to be a rectangular domain, owing to the fact that usually solutions are obtained by separation of variables as $\psi(R,Z) = f(R)g(Z)$ in which case the zeros of the specific functions $f(R)$ and $g(Z)$ can be used to let ψ vanish on the rectangular domain. Note that strictly speaking only the non-vacuum part of ψ has to vanish there.

The obtained analytical solutions discussed below all satisfy $p = p_0 - A\psi - \frac{1}{2}A'\psi^2$ and $F = F_0 - B\psi - \frac{1}{2}B'\psi^2$ such that $-\mu_0 R J_\phi = (A + A')R^2 + B + B'\psi$ with a varying number of the constants A, A', B and B' non-zero. Note that in these cases the GS-equation becomes an inhomogeneous linear PDE. Obviously the more free constants the source terms contain, the more control can be gained over the shape and characteristics of the solution.

4.1 Linear source terms (Solovév)

The classical Solovév solution uses $A' = B' = 0$ to obtain a right hand side $-\mu_0 R J_\phi = AR^2 + B$. This solution is equivalent to Hill's spherical vortex solution from fluid mechanics, with ψ representing the Stokes stream function.

The solution is given in Ref. [8]

$$\psi = \psi_h + \frac{A}{8} R^4 + \frac{B}{2} Z^2 \tag{23}$$

where ψ_h is the homogeneous solution. Note that the constant B on the right hand side of the GS-equation could also have been provided by a particular solution depending on R only, yielding $\psi_p = c_1 + c_2 R^2 + (B/2)R^2 \ln R + AR^4/8$, but obviously the above choice is the more convenient one, with A and B of equal sign allowing for closed flux surfaces.

An interesting aspect of this source function is that the homogeneous solution is a vacuum solution. We have already discussed the general vacuum solution in terms of Bessel functions. The homogeneous solution ψ_h can alternatively be provided based on the expansion

$$\psi_h = \sum_{n=0,2,..} f_n(R)Z^n \text{ with } R\frac{d}{dR}\left(\frac{1}{R}\frac{df_n(R)}{dR}\right) = -(n+1)(n+2)f_{n+2} \quad (24)$$

where only even powers of n are used to ensure top-down symmetry. Truncating the series for $n \geq 4$ yields

$$\psi_h = c_1 + c_2R^2 + c_3(R^4 - 4R^2Z^2) + c_4\left[R^2\ln(R) - Z^2\right] \quad (25)$$

with four degrees of freedom allowing one to prescribe the minor radius, aspect ratio, elongation and triangularity. A term linear in R is usually not considered. Scaling the solution with a constant factor yields a new solution with the same shape but with a linearly rescaled value of the plasma current.

A specific form of the Solovév solution is given in Ref. [7]

$$\psi = (C + DR^2)^2 + \frac{1}{2}\left[B + (A - 8D^2)R^2\right]Z^2 \quad (26)$$

with C and D integration constants. Note that $\Delta^*\psi$ yields $8DR^2$ from the first term and $B+(A-8D^2)R^2$ from the second, thereby reproducing the source term $A+BR^2$. Redefining the integration constants one can rewrite the solution in terms of the dimensionless coordinates $x = (R-R_0)/a = r - \epsilon^{-1}$ and $y = Z/a$

$$\bar{\psi} = \left[x - (\epsilon/2)(1 - x^2)\right]^2 + (1 - \epsilon^2/4)\left[1 + \tau\epsilon x(2 + \epsilon x)\right]\frac{y^2}{\sigma^2} \quad (27)$$

with $\bar{\psi} = 1$ going through $(x = \pm 1, y = 0)$ and $(x = 0, y = \pm\sigma)$ and τ some measure of the triangularity. The shift of the magnetic axis is then given by $(\sqrt{1+\epsilon^2} - 1)\epsilon$. The 'eigenvalues' A and B are given by

$$A = \frac{2}{\sigma^2}(1 - \epsilon^2/4 + \sigma^2) \quad (28)$$

$$B = 2\epsilon(\tau(1 - \epsilon^2/4) + \sigma^2)/(1 - \epsilon^2/4 + \sigma^2) \quad (29)$$

4.2 Quadratic source terms

With $\partial p/\partial\psi = -A'\psi$ and $\partial\frac{1}{2}F^2/\partial\psi = -B'\psi$ one has as a source term $RJ_\phi = (A'R^2 + B')\psi$. Note that in this case the homogeneous solution is no longer the vacuum solution. Inserting the assumption $\psi(R, Z) = f(R)\cos kz$ into the GS-equation yields for the R-equation

$$R\frac{d}{dR}\left(\frac{1}{R}\frac{df}{dR}\right) = (A'R^2 + B' + k^2)f \quad (30)$$

Introducing the coordinate transformation $\rho = cR^2$ such that $\frac{d}{dR} = 2cR\frac{d}{d\rho}$ we obtain

$$4c\rho\frac{d^2f}{d\rho^2} = \left(\frac{A'\rho}{c} + (B'-k^2)\right) f \text{ or } \frac{d^2f}{d\rho^2} + \left(\frac{-A'}{4c^2} - \frac{(B'-k^2)}{4c\rho}\right) f = 0 \quad (31)$$

which is the confluent hyper-geometric equation, see appendix C. E.g. with $c = \sqrt{-A'/4}$ the Coulomb wave equation, Eq. 63 for $l = 0$, is obtained

$$\frac{d^2f}{d\rho^2} + \left(1 - \frac{2\eta}{\rho}\right) f = 0 \quad (32)$$

where $\eta = (B' - k^2)/8c$ so that the solution becomes

$$\psi = \alpha \left[F_L(\eta,\rho) + \beta G_L(\eta,\rho)\right] \cos kz \quad (33)$$

Note that with $c = \sqrt{A'}$ and $\kappa = (k^2 - B')/4\sqrt{A'}$ the Whittaker equation Eq. 60 with $\mu = 1/2$ is obtained.

A linear superposition with different k_n and η_m is also possible, but these should have the same eigenvalue $B'_{mn} = B'$ as can be verified by substitution.

4.3 Dissimilar source functions

With $\partial p/\partial\psi = -A$ and $\partial\frac{1}{2}F^2/\partial\psi = -B - B'\psi$ one has as a right hand side $-\mu_0 R J_\phi = AR^2 + B + B'\psi$ yielding

$$\Delta^*\psi - B'\psi = AR^2 + B \quad (34)$$

which has the particular solution $\psi_p = -(AR^2 + B)/B'$. Note that now the particular solution instead of the homogeneous solution is a vacuum solution, i.e. has $\Delta^*\psi = 0$. The homogeneous solution is found [6] by separation of variables $\psi_h = f(R)\cos kZ$ only keeping cosine terms and solving

$$R\frac{d}{dR}\left(\frac{1}{R}\frac{df}{dR}\right) + \nu f = 0 \quad (35)$$

with $k^2+\nu^2 = -B'$ in the plasma region and $k^2+\nu^2 = 0$ in the vacuum. The equation for $f(r)/R$ is Bessel's equation so that $f(R) = R\left(c_\nu J_1(\nu R) + d_\nu Y_1(\nu R)\right)$ and

$$\psi(R,Z) = -(AR^2+B)/B' + \sum_\nu R\left(c_\nu J_1(\nu R) + d_\nu Y_1(\nu R)\right)\cos k_\nu Z \quad (36)$$

Alternatively a summation over k could have been used as long as $k^2 + \nu^2 = -B'$ is satisfied over the region of the plasma. Note that apart from this condition the homogeneous solution is the same as the vacuum solution.

Assuming the current density vanishes on the (perfectly conducting) walls of a square domain from $R = R_0 \pm a$ to $Z = \pm b$ one can take $\psi_h = 0$ on the boundary because $\Delta^*\psi_p = 0$. We now choose $k_n = (n+\frac{1}{2})\pi/b$ to ensure $\psi_h(R,\pm b) = 0$, $d_{mn} = -c_{mn}J_1(\nu_m(R_0-a))/Y_1(\nu_m(R_0-a))$ to ensure $\psi_h(R_0-a,Z) = 0$ and we choose ν_m be the root of

$$\frac{J_1(\nu_m(R_0+a))}{J_1(\nu_m(R_0-a))} = \frac{Y_1(\nu_m(R_0+a))}{Y_1(\nu_m(R_0-a))} \tag{37}$$

to ensure $\psi_h(R_0+a,Z) = 0$. Now $-B' = (n+\frac{1}{2})^2\pi^2/b^2 + \nu_m^2$ such that

$$\begin{aligned}\psi(R,Z) &= \sum_{m,n} c_{mn}R\left[J_1(\nu_m R) - \frac{J_1(\nu_m(R_0-a))}{Y_1(\nu_m(R_0-a))}Y_1(\nu_m R)\right]\cos\pi(n+\frac{1}{2})\frac{Z}{b} \\ &\quad + \frac{AR^2+B}{(n+\frac{1}{2})^2\pi^2/b^2+\nu_m^2} \end{aligned}\tag{38}$$

An alternative approach is pursued in Ref. [9], in which both the homogeneous and the particular solution are expanded as $\psi(R,Z) = R\sum_{m,n} c_{mn}G_{mn}(R,Z))$ and $-(AR^2+B) = R\sum_{m,n} f_{mn}G_{mn}(R,Z)$ where the 'eigenfunctions' $G_{mn}(R,Z)$ are the cosine-Bessel combination used in the above solution. The criterion $k_m^2 + \nu_n^2 = -B'$ is relaxed to $k_m^2 + \nu_n^2 = -B'_{mn}$ resulting after substitution in $c_{mn} = f_{mn}/(B'_{mn} - B')$. In this way the entire solution can be written in terms of the same basis functions G_{mn} as

$$\psi = R\sum_{m,n}\frac{f_{mn}}{B'_{mn}-B'}\left[J_1(\nu_m R) - \frac{J_1(\nu_m(R_0-a))}{Y_1(\nu_m(R_0-a))}Y_1(\nu_m R)\right]\cos\pi(n+\frac{1}{2})\frac{Z}{b} \tag{39}$$

where f_{mn} are the expansion coefficients of $-(AR^2+B)$, but the B'_{mn} can be chosen freely. Note that this is just the general vacuum solution Eq. 16 but without the freedom in the coefficient d_n of Y_1.

4.4 The most general linear solution

The most elaborate solution available is given in Ref. [1] for arbitrary coefficients A, B, A', B' and is expressed in terms of hyper geometric functions.

4.5 Nonlinear solutions

In Ref. [3] it is shown by some clever transformations that

$$\psi = -\frac{6}{9aR^2 + k_1(z+c_1)^2} \text{ with } k_2 = 2a(k_1 - 9a) \tag{40}$$

and

$$\psi = \frac{\sqrt{2}}{\sqrt{-9aR^2 + \alpha^2(z+c_2)^2}} \text{ with } \beta = -\frac{27}{4}a(\alpha^2+9a) \tag{41}$$

are solutions to the Grad-Shafranov equation with right hand side $-\mu_0 R J_\phi = -\psi^2(k_1 + k_2 R^2 \psi)$ and $-\mu_0 R J_\phi = -\psi^3(\alpha^2 - \beta R^2 \psi^2)$. Given the experimental obervation of sharp gradients in the physical quantities near the plasma edge, the dependence of p and F on a high power of ψ is desirable. The use of these solutions is however limited because the shapes described by these solutions does not resemble that of a tokamak. For top-down symmetric configurations $c_1 = c_2 = 0$ such that there are, apart from the variables associated with p and F, also no free parameters that can be used to control the shape of these solutions.

A GS in straight cylinder

A.1 Derivation GS-equation in a cylinder

In a straight cylindrical geometry, a derivation analogous to the one used in a toroidal geometry can be used to rewrite the force balance between the pressure force and the Lorentz force under the assumption $\partial/\partial z = 0$. Denoting the z-component of the vector potential with A

$$\mathbf{B} = \nabla A \times \hat{\mathbf{z}} + B_z \hat{\mathbf{z}} \tag{42}$$

Splitting the Lorentz force into two parts $\mathbf{J} \times \mathbf{B} = (j_z \hat{\mathbf{z}}) \times \mathbf{B}_\perp + \mathbf{j}_\perp \times (B_z \hat{\mathbf{z}})$. Using Ampere's law and some straightforward vector operations $\mu_0 \mathbf{J} = \nabla \times B = -\nabla^2 A \hat{z} + \nabla B_z \times \hat{\mathbf{z}}$ so that the force balance becomes

$$\begin{aligned} \nabla p &= \frac{1}{\mu_0} \left[(-\nabla^2 A \hat{\mathbf{z}}) \times (\nabla A \times \hat{\mathbf{z}}) + (\nabla B_z \times \hat{\mathbf{z}}) \times (B_z \hat{\mathbf{z}}) \right] && (43) \\ &= \frac{-1}{\mu_0} \left[(\nabla^2 A) \nabla A + B_z \nabla B_z \right] && (44) \end{aligned}$$

Rearranging the cross-products above, we see that that

$$\hat{\mathbf{z}} \times \mathbf{B}_\perp = \nabla A \tag{45}$$

With $\nabla p = (dp/dA)\nabla A$ and $\nabla B_z = (dB_z/dA)\nabla A$ we arrive at the Grad-Shafranov equation for a straight cylinder

$$\nabla^2 A = -\mu_0 \frac{d}{dA} (p + \frac{B_z^2}{2\mu_0}) \tag{46}$$

Using Ampere's law $\mathbf{J} = \mu_0^{-1} \nabla \times B = 0$ one can rewrite the Lorentz force as $-\nabla B^2 + \mathbf{B} \cdot \nabla \mathbf{B}$ such that the radial component of the force balance in cylindrical coordinates is written

$$\frac{\partial}{\partial r} \left(p + \frac{B^2}{2\mu_0} \right) = \frac{1}{\mu_0} \left(B_r \frac{\partial}{\partial r} + \frac{B_\theta}{r} \frac{\partial}{\partial \theta} + B_z \frac{\partial}{\partial z} \right) B_r - \frac{B_\theta^2}{\mu_0 r} \tag{47}$$

We can distinguish two main simplifying cases

A.2 θ-pinch

With $p = p(r)$, $\mathbf{B} = B_z(r) \hat{\mathbf{z}}$ and $\mathbf{J} = J_\theta(r) \hat{\theta}$ the force balance becomes

$$\frac{d}{dr} \left(p + \frac{B_z^2}{2\mu_0} \right) = 0 \tag{48}$$

which can be integrated to yield $p(r) + B_z^2(r)/2\mu_0 = B_0^2/2\mu_0$

A.3 z-pinch

with $p = p(r)$, $\mathbf{B} = B_z\theta(r)\hat{\theta}$ and $\mathbf{J} = J_z(r)\hat{\mathbf{z}}$ the force balance becomes

$$\frac{d}{dr}\left(p + \frac{B_\theta^2}{2\mu_0}\right) + \frac{B_\theta}{\mu_0 r} = 0 \tag{49}$$

which can be integrated by assuming some current distribution $J_z(r)$ and calculating the associated magnetic field $B_\theta(r)$ from Ampere's law. A parabolic current distribution e.g. leads to a magnetic field increasing from the centre but towards the plasma edge is decreasing again due to the fact that the total enclosed current increases sub-linearly with radius. This means that the magnetic pressure force will be in the outward radial direction and that the magnetic tension will have to act to confine the plasma.

B Alternative formulations

B.1 Spherical coordinates

In spherical coordinates $\mathbf{B} = \nabla\psi \times \nabla\phi + Q\nabla\phi = \frac{1}{r\sin\theta}(\frac{1}{r}\frac{\partial\psi}{\partial\theta}, -\frac{\partial}{\partial r}, Q)$ with $p(\psi)$ and $p(\psi)$ flux functions [5]

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r^2}\sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right) + Q\frac{\partial Q}{\partial\psi} + \mu r^2\sin^2\theta\frac{\partial p}{\partial\psi} = 0 \tag{50}$$

B.2 Dimensionless local coordinates

Nondimensionalizing the coordinates $r = R/a$ and $y = Z/a$, introducing a unit flux variable $\bar{\psi} = \psi/\psi_1$ and scaling the source terms with the vacuum magnetic field B_0 at $R = R_0$, i.e. $p' = p/B_0^2$ and $\frac{1}{2}f'^2 = \frac{1}{2}(F^2 - F_\infty^2)/B_0^2a^2$ with $F_\infty \equiv R_0B_0$ the Grad-Shafranov equation is written in dimensionless form

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\bar{\psi}}{\partial r}\right) + \frac{\partial^2\bar{\psi}}{\partial z^2} = -\alpha^2\frac{d\frac{1}{2}f'^2}{d\bar{\psi}} - r^2\alpha^2\frac{dp'}{d\bar{\psi}} \tag{51}$$

where the only dimensionless number $\alpha^2 = (B_0a^2/\psi_1)^2$ can be incorporated into p' and f' and as such can be used to rescale the solution. This generally occurs for linear equations. For high beta tokamaks the pressure scales as ϵ and F^2 as ϵ^{-1}, where $\epsilon \equiv a/R_0$ is the inverse aspect ratio. In order to have quantities of order one [2] we take $p' = \alpha^2p/\epsilon B_0^2$ and $f'^2 = \alpha^2\epsilon F^2/B_0^2a^2$. Now we notice that the sum of these two profile functions

$$p' + \frac{1}{2}f'^2 = \frac{\alpha^2}{2\epsilon B_0^2}\left(2p + \left(\frac{\epsilon}{a}\right)^2(F^2 - F_\infty^2)\right) = \frac{\alpha^2}{2\epsilon B_0^2}\left(2p + \left(\frac{R}{R_0}\right)^2(B_\phi^2 - B_0^2)\right) \tag{52}$$

For a straight cylindrical θ-pinch, we found $2p = -(B_\phi^2 - B_0^2)$ such that $p' + \frac{1}{2}f'^2 = \mathcal{O}((\alpha^2/2\epsilon B_0^2)\epsilon^2B_0^2) = \mathcal{O}(\epsilon\alpha^2/2)$. The shape of the two functions p' and $\frac{1}{2}f'^2$ thus do not differ to order ϵ. Therefore a new quantity of order unity can be introduced $g' \equiv \epsilon^{-1}(p' + \frac{1}{2}f'^2)$ measuring the deviation from a θ-pinch. Finally unit profiles are obtained by introducing $\Gamma' = g'/A$ and $\Pi' = 2p'/AB$ with $A = g'(0)$ and $p'(0) = \frac{1}{2}AB$ and re-shifting them using their values Γ'_1 and Π'_1 at $\bar{\psi} = 1$: $\Gamma = (\Gamma' - \Gamma'_1)/(1 - \Gamma'_1)$ and $\Pi = (\Pi' - \Pi'_1)/(1 - \Pi'_1)$ to obtain genuine unit profiles. With $x = (R - R_0)/a$ and $y = Z/a$ the Grad-Shafranov equation then becomes

$$\frac{\partial^2\bar{\psi}}{\partial x^2} - \frac{\epsilon}{1 + \epsilon x}\frac{\partial\bar{\psi}}{\partial x} + \frac{\partial^2\bar{\psi}}{\partial y^2} = A\left[\Gamma(\bar{\psi}) + Bx(1 + \frac{1}{2}\epsilon x)\Pi(\bar{\psi})\right] \tag{53}$$

C Special functions (from Wikipedia)

The *Hypergeometric series* is defined as a power series $\sum_n \beta_n z^n$ in which the ratio of successive coefficients is a rational function of n, i.e. $\frac{\beta_{n+1}}{\beta_n} = \frac{A(n)}{B(n)}$ where $A(n)$ and $B(n)$ are polynomials in n. Alternatively:

$$ {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{z^n}{n!} \tag{54}$$

where the rising factorial or Pochhammer symbol denotes $(a)_n = a(a+1)(a+2)...(a+n-1)$, $(a)_0 = 1$. Examples of functions that can be built up like this are e^z, $(1-z)^k$, $z^{-1}\log 1+z$, $z^{-1}\arcsin z$, $z^{-1}\arctan z$, etc.

An important special ODE is the hypergeometric differential equation

$$z(1-z)\frac{d^2w}{dz^2} + [c-(a+b+1)z]\,\frac{dw}{dz} - abw = 0 \tag{55}$$

whose two linearly independent solutions can be built from ${}_2F_1(a, b; c; z)$.. Every second-order linear ODE with three regular singular points can be transformed into this equation.

A *confluent hypergeometric function* is a solution of a confluent hypergeometric equation, which is a degenerate form of a hypergeometric differential equation where two of the three regular singularities merge (Lat. "confluere") into an irregular singularity. There exist three alternative formulations

Kummer's (confluent hypergeometric) function

$$z\frac{d^2w}{dz^2} + (b-z)\frac{dw}{dz} - aw = 0. \tag{56}$$

With linearly independent solutions $M(a, b, z)$ (Kummer's function (first kind)) and $U(a, b, z)$ (Kummer's function (second kind))

$$M(a, b, z) = {}_1F_1(a; b; z) = \Phi(a, b, z) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(b)_n n!} \tag{57}$$

$$U(a, b, z) = \frac{\pi}{\sin \pi b}\left(\frac{M(a, b, z)}{\Gamma(1+a-b)\Gamma(b)} - z^{1-b}\frac{M(1+a-b, 2-b, z)}{\Gamma(a)\Gamma(2-b)}\right) \tag{58}$$

and has the formal expansion as a power series (which converges nowhere)

$$U(a, b, z) = z^{-a} \cdot {}_2F_0\left(a, 1+a-b; ; -\frac{1}{z}\right). \tag{59}$$

Whittaker's equation

$$\frac{d^2w}{dz^2} + \left(-\frac{1}{4} + \frac{\kappa}{z} + \frac{1/4 - \mu^2}{z^2}\right) w = 0. \tag{60}$$

With as solution the Whittaker functions $M_{,}(z)$, $W_{,}(z)$, defined in terms of Kummer functions M and U by

$$M_{\kappa,\mu}(z) = e^{-z/2} z^{\mu+1/2} M(\mu - \kappa + 1/2, 1 + 2\mu, z) \tag{61}$$

$$W_{\kappa,\mu}(z) = e^{-z/2} z^{\mu+1/2} U(\mu - \kappa + 1/2, 1 + 2\mu, z) \tag{62}$$

Coulomb wave equation

$$\frac{d^2w}{d\rho^2} + \left(1 - \frac{2\nu}{\rho} - \frac{l(l+1)}{\rho^2}\right) w = 0 \tag{63}$$

where l is usually a non-negative integer. The solutions are called Coulomb wave functions F_L and G_L. Putting $x = 2i\rho$ changes the Coulomb wave equation into the Whittaker equation, so Coulomb wave functions can be expressed in terms of Whittaker functions with imaginary arguments.

References

[1] C.V.Atanasiu, S.Gunter, K.Lackner, and I.G.Miron. Analytical solutions to the Grad-Shafranov equation. *Physics of Plasmas*, 11(7):3510, 2004.

[2] J.P.Goedbloed. Some remarks on computing axisymmetric equilibria. *Computer Physics Communications*, 31:123, 1984.

[3] A. H. Khater and S. M. Moawad. Exact solutions for axisymmetric nonlinear magnetohydrodynamic equilibria of aligned magnetic field and plasma flow with applications to astrophysics and plasma confinement devices. *Physics of Plasmas*, 16(052504), 2009.

[4] K.H.Tsui. Computation of mhd equilibrium of tokamak plasma. 93:1, 1991.

[5] K.H.Tsui. Toroidal equilibria in spherical coordinates. *Physics of Plasmas*, 15:112506, 2008.

[6] Lai and Verleun. An exact solution for toroidal plasma in equilibrium. 19(7):1066, 1976.

[7] R.Keppens and J.W.S.Blokland. Computing ideal magnetohydrodynamic equilibria. 46(2), 2006.

[8] S.B.Zheng, A.J.Wootton, and E.R.Solano. Analytical tokamak equilibrium for shaped plasmas. 3:1176, 1996.

[9] S.Wang and J.Yu. An exact solution of the Grad-Shafranov-Helmhotz equation with central current density reversal. *Physics of Plasmas*, 12(062501), 2005.